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Direct Gauging of the Poincaré Group V. Group Scaling, Classical Gauge Theory, and Gravitational Corrections

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Homogeneous scaling of the group space of the Poincaré group, P_{10}, is shown to induce scalings of all geometric quantities associated with the local action of P_{10}. The field equations for both the translation and the Lorentz rotation compensating fields reduce to $O(1)$ equations if the scaling parameter is set equal to the general relativistic gravitational coupling constant $8\pi Gc^{-4}$. Standard expansions of all field variables in power series in the scaling parameter give the following results. The zeroth-order field equations are exactly the classical field equations for matter fields on Minkowski space subject to local action of an internal symmetry group (classical gauge theory). The expansion process is shown to break P_{10}-gauge covariance of the theory, and hence solving the zeroth-order field equations imposes an implicit system of P_{10}-gauge conditions. Explicit systems of field equations are obtained for the first- and higher-order approximations. The first-order translation field equations are driven by the momentum-energy tensor of the matter and internal compensating fields in the zeroth order (classical gauge theory), while the first-order Lorentz rotation field equations are driven by the spin currents of the same classical gauge theory. Field equations for the first-order gravitational corrections to the matter fields and the gauge fields for the internal symmetry group are obtained. Direct Poincaré gauge theory is thus shown to satisfy the first two of the three-part acid test of any unified field theory. Satisfaction of the third part of the test, at least for finite neighborhoods, seems probable.

1. INTRODUCTION

A direct gauge theory for the Poincaré group was given in part I of this work (Edelen, 1985a) by implementing a minimal replacement operator based on a realization of P_{10} as a subgroup of $GL(5, R)$ that mapped an affine set into itself. This theory was extended in II (Edelen, 1985b) and

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the problem of gauging a Poincaré-invariant theory with an internal symmetry group was examined in III (Edelen, 1985c). Representations for the connection and curvature structures were obtained in IV (Edelen, 1985d), and a free field Lagrangian for the local action of P_{10} was specified. Explicit citation of equations from these four papers will be made by hyphenation with the appropriate Roman numeral.

The gauge theory that obtains by this procedure works whenever the corresponding gauge theory of matter fields on Minkowski space-time with an internal symmetry group is well defined. It exhibits the correct gravitational limit in that it reduces to general relativity whenever the total momentum-energy tensor of the matter fields and the internal symmetry-compensating fields are symmetric and the spin currents all vanish. Further, the holonomy group of the resulting space-time U_4 is the lifted component of the Lorentz group that is connected to the identity, the representations contain contributions from both the Lorentz and the translation sectors, the torsion is algebraically determined by the spin currents and the coframe fields, and the field equations for the compensating fields of P_{10} are first-order equations without differential coupling.

Although these results argue in favor of a unified gauge field theory, there is an acid test that this and any other theory must pass before its presumption of correctness may be tentatively accepted. We know that gauge theories of matter fields with internal symmetry groups on Minkowski space-time lead to very accurate predictions in many situations [e.g., quantum electrodynamics with the $U(1)$ gauge group, which is the simplest and probably the most accurate]. Since the underlying manifold is Minkowski space-time, gravitational effects are necessarily ignored (switched off). Now, gauging the Poincaré group by allowing it to act locally has the effect of switching gravitational effects back on, and the theory given in I–IV shows that the effects of this process influence the matter fields and the compensating fields for local action of the internal symmetry group. Accordingly, local action of the Poincaré group will change the matter fields and the compensating fields for the internal symmetry group relative to the evaluations computed in the Minkowski space formulation. There will therefore be changes in the observables associated with the matter fields and the compensating fields for the internal symmetry group. The acid test consists in showing that the changes in the predicted values of observables fall within the experimental error whenever the Minkowski-space formulation is in good agreement with experiment.

There are three aspects of the acid test that require analysis. A clear and pressing first requirement is that the gauge theory reduce to the Minkowski-space formulation in an appropriate asymptotic limit. This requirement is shown to be satisfied in Section 6. Second, field equations for the fields that correct the Minkowski-space evaluations have to be obtained. Section 7 acquits this task. These two of the three parts can and have been solved in the general context (i.e., without specification of the Lagrangian for the matter fields or the compensating fields for the internal symmetry group). The third part is necessarily specific to explicit situations, since it requires calculation of changes in observables that obtain as a consequence of switching on gravity. This part is left to those better qualified in the specific tasks. I can only remark that the correction terms obtain as first-order terms in an expansion in the general relativistic coupling constant $8\pi G/c^4$, which is approximately 10^{-40} in atomic units. It may thus be anticipated that changes in the observables will be "small," barring unforseen resonances or unreasonable pathologies associated with solutions of the correction fields.

2. GROUP SCALING

The unit in the group space of the Poincaré group has been assigned in a default manner to be unity. It is clear, however, that we can always change the unit in the group space of P_{10} if this should prove useful; simply note that any multiple of a basis for a Lie algebra is another basis for that Lie algebra. Thus, if we use ε to denote the new group space unit, the generators of P_{10} undergo the transitions

$$\mathbf{l}_\alpha \to \bar{\mathbf{l}}_\alpha = \varepsilon \mathbf{l}_\alpha, \qquad \mathbf{e}_i \to \mathbf{e}_i = \varepsilon \mathbf{e}_i \tag{1}$$

and

$$C_\alpha{}^\beta{}_\gamma \to \bar{C}_\alpha{}^\beta{}_\gamma = \varepsilon C_\alpha{}^\beta{}_\gamma \tag{2}$$

These transitions entail corresponding transitions in all of the underlying P_{10} quantities. The distortion 1-forms (coframe fields) are given by (I-12), and hence their coefficients may be written in the equivalent form

$$B^i_j = \delta^i_j + W^\alpha_j l_\alpha{}^i{}_k x^k + \phi^k_j e^i_k$$

where the W's are the components of the compensating 1-forms for the local action of the Lorentz sector and the ϕ's are the components of the compensating 1-forms for the local semidirect product action of the translation sector. We therefore have the transition

$$B^i_j \to \bar{B}^i_j = \delta^i_j + \varepsilon (W^\alpha_j l_\alpha{}^i{}_k x^k + \phi^i_j) \tag{3}$$

The corresponding transition of the coefficients of the frame fields will be written

$$b^i_j \to \bar{b}^i_j = \delta^i_j + \varepsilon \bar{\bar{b}}^i_j \tag{4}$$

since

$$b^i_j B^j_k = \delta^i_j$$

It therefore follows from (IV-13) that

$$L_{\alpha i}^{\ \ j} \to \bar{L}_{\alpha i}^{\ \ j} = \varepsilon \bar{\bar{L}}_{\alpha i}^{\ \ j}, \qquad \bar{\bar{L}}_{\alpha i}^{\ \ j} = \bar{b}^i_p l_{\alpha q}^{\ \ p} \bar{B}^q_j \tag{5}$$

The group scaling of P_{10} obviously induces transitions in the connection coefficients. The anholonomic connection coefficients are given by (IV-28),

$$\tilde{\Gamma}^i_{kj} = W^\alpha_k l_{\alpha j}^{\ \ i}$$

and hence we have the transition

$$\tilde{\Gamma}^i_{kj} \to \bar{\gamma}^i_{kj} = \varepsilon W^\alpha_k l_{\alpha j}^{\ \ i} = \varepsilon \tilde{\Gamma}^i_{kj} \tag{6}$$

Since the connection coefficients for the adjoint action of P_{10} have the evaluation

$$\Gamma_{k\alpha}^{\ \ \beta} = W^\rho_k C_\rho^{\ \beta}{}_\alpha$$

we have

$$\Gamma_{k\alpha}^{\ \ \beta} \to \bar{\Gamma}_{k\alpha}^{\ \ \beta} = \varepsilon \Gamma_{k\alpha}^{\ \ \beta} = \varepsilon W^\rho_k C_\rho^{\ \beta}{}_\alpha \tag{7}$$

The total connection coefficients for the matter fields are given by [see (I-67) and (III-23)]

$$\Gamma_{kA}^{\ \ B} = W^\alpha_k M_{\alpha A}^{\ \ B} + A^b_j f_{bA}^{\ \ B}$$

Here, the A's are the components of the compensating 1-forms for the local action of G_r and the f's are a basis for the Lie algebra of the action of G_r on the matter fields. These connection coefficients thus undergo the transition

$$\Gamma_{kA}^{\ \ B} \to \bar{\Gamma}_{kA}^{\ \ B} = \varepsilon W^\alpha_k M_{\alpha A}^{\ \ B} + A^b_k f_{bA}^{\ \ E} \tag{8}$$

where the M_α are a basis for the Lie algebra of P_{10} on the matter field representation. The ε in (8) comes from the fact that scaling in the group space of P_{10} necessarily carries over to any representation of P_{10}, and hence we have the transition $\mathbf{M}_\alpha \to \varepsilon \mathbf{M}_\alpha$. On the other hand, scaling in the group space of P_{10} leaves the group space of the internal symmetry group unchanged, so that there is no ε in the terms that come from the local action of G_r.

The holonomic connection coefficients are given by (IV-12),

$$\Gamma^i_{kj} = W^\alpha_k L_{\alpha j}^{\ \ i} + b^i_p \partial_k B^p_j$$

Thus, since

$$B_j^p = \delta_j^p + W_j^\alpha l_{\alpha k}^p x^k + \phi_j^p$$

we have the transition

$$\Gamma_{kj}^i \to \bar{\Gamma}_{kj}^i = \varepsilon \bar{\bar{\Gamma}}_{kj}^i = \varepsilon (W_k^\alpha \bar{\bar{L}}_{\alpha j}^{\ \ i} + \bar{b}_p^i \, \partial_k B_j^p) \tag{9}$$

The torsion thus undergoes the transition

$$S_{kj}^i \to \bar{S}_{kj}^i = \varepsilon \bar{\bar{S}}_{kj}^i = \varepsilon \bar{\bar{\Gamma}}_{[kj]}^{\ i} \tag{10}$$

The induced transition of the curvature quantities are now easily obtained. It follows directly from (I-21) that the $L(4, R)$ curvature 2-forms undergo the transition

$$\theta^\alpha \to \bar{\theta}^\alpha = dW^\alpha + \tfrac{1}{2} \varepsilon C_\gamma{}^\alpha{}_\beta W^\gamma \wedge W^\beta \tag{11}$$

Thus, since the holonomic curvature tensor has the evaluation (IV-29),

$$R_{kmj}^{\ \ \ i} = \theta_{km}^\alpha L_{\alpha j}^{\ \ i}$$

we have

$$R_{kmj}^{\ \ \ i} \to \bar{R}_{kmj}^{\ \ \ i} = \varepsilon \bar{\bar{R}}_{kmj}^{\ \ \ i} = \varepsilon \bar{\theta}_{km}^\alpha \bar{\bar{L}}_{\alpha j}^{\ \ i} \tag{12}$$

It now easily follows that

$$g_{ij} \to \bar{g}_{ij} = \bar{B}_i^p h_{pq} \bar{B}_j^q, \qquad g^{ij} \to \bar{g}^{ij} = \bar{b}_p^i h^{pq} \bar{b}_q^j \tag{13}$$

and hence (12) gives

$$R_{ij} \to \bar{R}_{ij} = \varepsilon \bar{\bar{R}}_{ij} = \varepsilon \bar{\bar{R}}_{kij}^{\ \ \ k}, \qquad R \to \bar{R} = \varepsilon \bar{\bar{R}} \tag{14}$$

3. DETERMINATION OF THE SCALING PARAMETER

Up to this point, the value of ε is arbitrary. There is, however, a natural choice for ε, which I now proceed to demonstrate. The field equations for the P_{10} compensating fields, with the choice of the free field Lagrangian given in IV, are

$$R_{ij} - \tfrac{1}{2} R g_{ij} + \lambda g_{ij} = \kappa \tau_{ij} \tag{15}$$

$$g^{km} (S_{im}^j L_{\alpha k}^{\ \ i} - 2 S_{im}^i L_{\alpha k}^{\ \ j}) = \frac{\kappa}{B} J_\alpha^j \tag{16}$$

[see (IV-57) and (IV-67)]. Here, τ_{ij} are the components of the total momentum-energy tensor (matter+internal symmetry fields), κ is the general relativistic coupling constant $8\pi G/c^4$, and the J's are the spin currents that are given by

$$J_\alpha^i = b_k^i L_A^k M_{\alpha B}^{\ A} \Psi^B \tag{17}$$

Under group scaling, the occurrence of the M_α in (17) shows that

$$J^i_\alpha \to \bar{J}^i_\alpha = \varepsilon \bar{\bar{J}}^i_\alpha = \varepsilon \bar{b}^i_k \bar{L}^k_A M_{\alpha B}^{\ A} \Psi^B \tag{18}$$

and hence (15) and (16) go over into

$$\varepsilon(\bar{\bar{R}}_{ij} - \tfrac{1}{2}\bar{\bar{R}}\bar{g}_j) + \lambda \bar{g}_{ij} = \kappa \bar{\tau}_{ij} \tag{19}$$

$$\varepsilon^2 \bar{g}^{km}(\bar{\bar{S}}^j_{im} \bar{\bar{L}}_{\alpha k}^{\ i} - 2\bar{\bar{S}}^i_{im} \bar{\bar{L}}_{\alpha k}^{\ j}) = \varepsilon \frac{\kappa}{B} \bar{\bar{J}}^i_\alpha \tag{20}$$

Thus, the field equations given by (19) for the translation compensating fields are $O(\varepsilon)$ if $\lambda = O(\varepsilon)$, while the field equations for the Lorentz compensating fields are $O(\varepsilon^2)$.

Now, all of the barred and double-barred quantities in these equations are $O(1)$ relative to ε. Accordingly, (19) and (20) become simultaneously $O(1)$ equations for the choice

$$\varepsilon = \kappa = 8\pi G c^{-4} \tag{21}$$

We thus have

$$\bar{\bar{R}}_{ij} - \tfrac{1}{2}\bar{\bar{R}}\bar{g}_{ij} + \frac{\lambda}{\kappa} \bar{g}_{ij} = \bar{\tau}_{ij} \tag{22}$$

$$\bar{g}^{km}(\bar{\bar{S}}^j_{im} \bar{\bar{L}}_{\alpha k}^{\ i} - 2\bar{\bar{S}}^i_{im} \bar{\bar{L}}_{\alpha k}^{\ j}) = \frac{1}{B} \bar{\bar{J}}^i_\alpha \tag{23}$$

for the choice of ε given by (21).

Although we could have taken $\kappa = k\varepsilon$ with $k = O(1)$ instead of (21), there is no real purpose served. On the other hand, there is no loss of generality in making the choice (21), because of the freedom in the choice of ε, and it does account for the coupling constant κ that occurs in the free field Lagrangian for P_{10}. The clincher is the fact that this choice of ε simultaneously reduces all P_{10} field equations to $O(1)$ equations.

4. SCALING OF THE FIELD EQUATIONS FOR MATTER AND INTERNAL SYMMETRY

Associated with the P_{10} field equations are field equations for the matter fields and for the compensating fields of the internal symmetry group G_r. The state variables of the matter fields are denoted by $\{\Psi^A: 1 \leq A \leq m\}$. Under minimal replacement for $P_{10} \times G_r$, the coordinate derivatives $\partial_i \Psi^A$ go over into the quantities

$$y^A_i = b^k_i(\partial_k \Psi^A + W^\alpha_k M_{\alpha E}^{\ A} \Psi^E + A^b_k f_{bE}^{\ A} \Psi^E) \tag{24}$$

It thus follows that the y's undergo the following transition when the group space of P_{10} is scaled:

$$y_i^A \rightarrow \bar{y}_i^A = \bar{b}_i^k(\partial_k \Psi^A + \varepsilon W_k^\alpha M_{\alpha E}^{\ A} \Psi^E + A_k^b f_{bE}^{\ A} \Psi^E) \tag{25}$$

The scaled y-fields are thus $O(1)$ fields relative to ε.

Action of the minimal replacement operator converts the Lagrangian $L(x^i, \Psi^A, \partial_i \Psi^A)$ for the matter fields on Minkowski space into the new Lagrangian $L(x^i, \Psi^A, y^A{}_i)$ on the space U_4. This new Lagrangian gives rise to the constitutive relations [see (III-22)]

$$L_A^i = \partial L/\partial y_i^A, \qquad L_A = \partial L/\partial \Psi \tag{26}$$

and hence scaling of the P_{10} group space gives

$$L_A^i \rightarrow \bar{L}_A^i = \partial \bar{L}/\partial \bar{y}_i^A, \qquad L_A \rightarrow \bar{L}_A = \partial \bar{L}/\partial \Psi^A \tag{27}$$

where

$$\bar{L} = L(x^i, \Psi^A, \bar{y}_i^A)$$

The matter field equations in U_4 have been shown to take the form (see III-23)

$$\partial_j \{B b_i^j L_A^i\} - \{W_j^\alpha M_{\alpha A}^{\ E} + A_j^b f_{bA}^{\ E})\{B b_i^j L_E^i\} = B L_A$$

with

$$B = \det(B_j^i) = (-g)^{1/2}$$

We thus have the P_{10}-scaled equations

$$\partial_j \{\bar{B} \bar{b}_i^j \bar{L}_A^i\} - (\varepsilon W_j^\alpha M_{\alpha A}^{\ E} + A_j^b f_{bA}^{\ E})\{\bar{B} \bar{b}_i^j \bar{L}_E^i\} = \bar{B} \bar{L}_A \tag{28}$$

The form of the scaled matter field equations, (28), although adequate for most purposes, does not exhibit the manifest gauge covariance that these equations possess. In order to bring out this aspect of the problem, let us introduce the notation

$$N_A^j = \bar{B} \bar{b}_i^j \bar{L}_A^i \tag{29}$$

It then readily follows that the N's are the components of a mixed tensor density of the indicated type. Accordingly, its total gauge-covariant derivative (see IV) is given by

$$\overset{T}{\nabla}_k N_A^j = \partial_k N_A^j + \bar{\Gamma}_{km}^j N_A^m - \bar{\Gamma}_{kA}^{\ E} N_E^j - \bar{\Gamma}_{km}^m N_A^j \tag{30}$$

and we have

$$\overset{T}{\nabla}_j N_A^j = \partial_j N_A^j - \bar{\Gamma}_{jA}^{\ E} N_E^j + 2 \bar{S}_{jm}^j N_A^m \tag{31}$$

It is now just a matter of combining (8), (29), and (31) in order to see that (28) assumes the equivalent form

$$(\overset{T}{\nabla}_j - 2\bar{S}^m_{mj})\{\bar{B}\bar{b}^j_i\bar{L}^i_A\} = \bar{B}\bar{L}_A \tag{32}$$

Now, B and the b's are gauge-covariant constant fields, and hence so are their P_{10} scaled values. We therefore have the simplified covariant form of the matter equations

$$\bar{b}^j_i(\overset{T}{\nabla}_j - 2\varepsilon\bar{\bar{S}}^m_{mj})\bar{L}^i_A = \bar{L}_A \tag{33}$$

when (10) is used. This form of the matter field equations has the additional advantage of showing the explicit coupling to the torsion field that was only implicit in the earlier version (28). In particular, we see that the torsion coupling terms are $O(\varepsilon)$!

Under the restriction that the free field Lagrangian for the gauge-compensating fields be at most quadratic in curvature and torsion, it was shown in III and IV that the total free field Lagrangian was the sum of a free field Lagrangian Π_G for G_r. Thus, the constitutive relations (III-25) assume the form

$$\tilde{G}^{ij}_b = BQ^{ij}_b, \qquad Q^{ij}_b = \partial(\Pi_G)/\partial\tilde{\theta}^b_{ij} \tag{34}$$

where the $\tilde{\theta}$'s are the components of the curvature 2-forms for the local action of the internal symmetry group. Scaling of the group space of P_{10} thus leads to the transition

$$\tilde{G}^{ij}_b \to \bar{B}\bar{Q}^{ij}_b, \qquad \bar{Q}^{ij}_b = \partial(\bar{\Pi}_G)/\partial\tilde{\theta}^b_{ij} \tag{35}$$

It is now simply a matter of substituting (35) into (III-26) in order to obtain the scaled field equations for the compensating fields for the local action of G_r:

$$\partial_j(\bar{B}\bar{Q}^{ji}_b) - A^a_j k_a{}^c{}_b \bar{B}\bar{Q}^{ji}_c = \tfrac{1}{2}\bar{B}\bar{b}^i{}_m \bar{L}^m_A f_{bE}^A \Psi^E \tag{36}$$

The k's in (36) are the structure constants for the internal symmetry group G_r.

A similar situation obtains here as with the matter field equations; namely, the equations represented by (36) are not displayed in a gauge-covariant form. In order to remedy this, we introduce the quantities

$$Y^{ji}_b = \bar{B}\bar{Q}^{ji}_b \tag{37}$$

which are the components of a mixed tensor density of the indicated type.

Total gauge-covariant differentiation gives

$$\overset{T}{\nabla}_k Y^{ji}_b = \partial_k Y^{ji}_b + \bar{\Gamma}^j_{km} Y^{mi}_b + \bar{\Gamma}^i_{km} Y^{jm}_b - \Gamma^c_{kb} Y^{ji}_c - \bar{\Gamma}^m_{km} Y^{ji}_b$$

The same procedure as that used with the matter field equations now gives

$$(\overset{T}{\nabla}_j - 2\varepsilon \bar{\bar{S}}^m_{mj})(\bar{B}\bar{Q}^{ji}_b) - \varepsilon \bar{\bar{S}}^i_{jm} \bar{B}\bar{Q}^{jm}_b = \tfrac{1}{2}\bar{B}\bar{b}^i_m \bar{L}^m_A f_{bE}{}^A \Psi^E \tag{38}$$

However, $\bar{B}$ is gauge-covariant constant, and hence (38) gives the simplified gauge-covariant constant field equations

$$(\overset{T}{\nabla}_j - 2\varepsilon \bar{\bar{S}}^m_{mj})\bar{Q}^{ji}_b - \varepsilon \bar{\bar{S}}^i_{jm} \bar{Q}^{jm}_b = \tfrac{1}{2}\bar{b}^i_m \bar{L}^m_A f_{bE}{}^A \Psi^E \tag{39}$$

The explicit coupling to the torsion field of U_4 is now exhibited, and it again turns out to be $O(\varepsilon)$. Note in particular that the same derivation operator occurs in the G_r-field equations as occurs in the matter field equations. There is a difference, however, for (39) entail the full torsion tensor in addition to its contraction, while the matter field equations couple only to the contracted torsion tensor.

5. EXPANSION IN THE GROUP SCALING PARAMETER

The evaluation of the scaling parameter obtained in Section 3, namely

$$\varepsilon = 8\pi G c^{-4}$$

shows that ε is very small except in systems of very exotic units ($\varepsilon \doteq 10^{-40}$ in atomic units). It is therefore natural to consider expansions of the various field quantities in power series ε:

$$\Psi^A = \Psi_0^A + \varepsilon \Psi_1{}^A + \varepsilon^2 \Psi_2{}^A + \cdots \tag{40}$$

$$A^b_j = A_{0j}{}^b + \varepsilon A_{1j}{}^b + \varepsilon^2 A_{2j}{}^b + \cdots \tag{41}$$

$$W^\alpha_i = W_{0i}{}^\alpha + \varepsilon W_{1i}{}^\alpha + \varepsilon^2 W_{2i}{}^\alpha + \cdots \tag{42}$$

$$\phi^i_k = \phi_{0k}{}^i + \varepsilon \phi_{1k}{}^i + \varepsilon^2 \phi_{2k}{}^i + \cdots \tag{43}$$

These expansions induce similar expansions of all quantities that occur in the field equations. For example, with the obvious notation, we have

$$\bar{L}^i_A = L_{0A}{}^i + \varepsilon L_{1A}{}^i + \varepsilon^2 L_{2A}{}^i + \cdots \tag{44}$$

Thus, quantities with a lower zero denote evaluation at $\varepsilon = 0$, quantities with a lower 1 denote evaluation of the derivative with respect to ε at $\varepsilon = 0$, etc.

The standard process is now at hand. We expand all terms in each field equation in ascending powers of ε and then equate corresponding terms (i.e., the equations are required to be satisfied as algebraic identities in ε). This leads to systems of field equations of zeroth order, of first order, of second order, etc. These are the subject of the next few sections.

There is one critical point that needs to be made here, however. The process of ε expansion of the field equations breaks the P_{10} gauge covariance of the theory. This is most easily seen by using (11) and

$$\bar{\theta}^{\alpha} = \theta_0{}^{\alpha} + \varepsilon\theta_1{}^{\alpha} + \varepsilon^2\theta_2{}^{\alpha} + \cdots$$

to obtain

$$\theta_0{}^{\alpha} = dW_0{}^{\alpha}$$

$$\theta_1{}^{\alpha} = dW_1{}^{\alpha} + \tfrac{1}{2}C_{\beta}{}^{\alpha}{}_{\gamma}W_0{}^{\beta} \wedge W_0{}^{\gamma}$$

$$\theta_2{}^{\alpha} = dW_2{}^{\alpha} + C_{\beta}{}^{\alpha}{}_{\gamma}W_0{}^{\beta} \wedge W_1{}^{\gamma}$$

A direct inspection shows that none of these resulting "ε-curvature" quantities is P_{10}-gauge-covariant.

A result of this nature should not be unexpected if classical gauge theory is to result in the zeroth-order approximation. Classical gauge theory obtains in Minkowski space, where P_{10} necessarily acts only globally. We are thus not at liberty to perform arbitrary local translations (arbitrary coordinate transformations) after the ε expansion has been made, because global P_{10} invariance does not imply local P_{10} invariance. In effect, the zeroth-order Minkoski-space approximation fixes the coordinate cover and hence there is an implied system of gauge conditions in operation.

6. THE MINKOWSKI-SPACE APPROXIMATION

We deal with the zeroth-order approximation in this section.

The formulas given in previous sections show that

$$B_{0}{}^{i}_{j} = b_{0}{}^{i}_{j} = \delta^{i}_{j}, \qquad B_0 = 1 \tag{45}$$

$$g_{0ij} = h_{ij}, \qquad g_0{}^{ij} = h^{ij} \tag{46}$$

$$R_{0}{}^{i}_{kmj} = 0, \qquad S_{0}{}^{i}_{jk} = 0 \tag{47}$$

Thus, U_4 devolves to Minkowski space in the zeroth-order approximation. Indeed, inspection of the previous formulas shows that all terms involving either the W's or the ϕ's vanish in the zeroth-order approximation.

Next, note that (25) implies

$$y_{0i}^{A}=\partial_i\Psi_0^{\;A}+A_{0i}^{\;b}f_{bE}^{\;A}\Psi_0^{\;E} \tag{48}$$

and hence minimal replacement for $P_{10}\times G_r$ reduces to minimal replacement for G_r in the zeroth-order approximation. It thus follows that

$$L_{0A}^{\;i},\; L_{0A},\; Q_{0b}^{\;ij}$$

coincide with what would obtain for a gauge theory on Minkowski space with the internal symmetry group G_r. Further, the only surviving field equations are the zeroth-order approximation equations that obtain from (28) and (36):

$$\partial_j L_{0A}^{\;j}-A_{0j}^{\;b}f_{bA}^{\;E}L_{0E}^{\;j}=L_{0A} \tag{49}$$

$$\partial_j Q_{0b}^{\;ji}-A_{0j}^{\;a}k_{a\;b}^{\;c}Q_{0c}^{\;ji}=\tfrac{1}{2}L_{0A}^{\;i}f_{bE}^{\;A}\Psi_0^{\;E} \tag{50}$$

These, however, are just the field equations for matter fields on Minkowski space subject to local action of the internal symmetry group G_r; that is, classical gauge field theory.

The zeroth-order approximation of direct gauge theory for $P_{10}\times G_r$ is the standard gauge theory for an internal symmetry group G_r on Minkowski space. I will therefore refer to the zeroth-order approximation as the Minkowski-space approximation. Direct gauge theory for the Poincaré group thus satisfies the first of the three parts of the acid test.

Now that we have classical gauge theory on Minkowski space-time as the zeroth-order approximation, the action of the Poincaré group is necessarily global. Thus, fixing the coordinate cover of M_4 necessarily inforces P_{10}-gauge conditions for the whole theory; that is, loss of gauge covariance in the ε-expansion process is an essential and unavoidable consequence of obtaining satisfaction of the first part of the acid test.

7. GRAVITATIONAL CORRECTIONS

We now know that the zeroth-order approximation is classical gauge theory on Minkowski space, and hence the action of the Poincaré group is necessarily global in the zeroth-order approximation. Accordingly, we may view the first- and higher order approximations as corrections that arise because of the local action of P_{10} (i.e., gravitational corrections). We confine our attention in this section to the first-order approximation in view of the value of ε established in Section 3.

The representations established in previous sections lead directly to the following evaluations:

$$B_{1j}^{\;i}=-b_{1j}^{\;i}=W_{0j}^{\;\alpha}l_{\alpha m}^{\;i}x^m+\phi_{0j}^{\;i} \tag{51}$$

$$B_1 = W_{0k}^{\alpha} l_{\alpha m}^{k} x^m + \phi_{0k}^{k} \tag{52}$$

$$g_{1ij} = h_{ip} B_{1j}^{p} + B_{1i}^{p} h_{pj} \tag{53}$$

$$g_1{}^{ij} = -h^{ip} g_{1pq} h^{qj} \tag{54}$$

$$\theta_0{}^{\alpha} = dW_0{}^{\alpha}, \qquad \theta_1{}^{\alpha} = dW_1{}^{\alpha} + \tfrac{1}{2} C_{\beta}{}^{\alpha}{}_{\gamma} W_0{}^{\beta} \wedge W_0{}^{\gamma} \tag{55}$$

Next, we note that (5), (12), and (14) imply

$$R_{1ij} = \theta_{0mi}^{\alpha} l_{\alpha j}^{m}, \qquad R_1 = \theta_{0mk}^{\alpha} l_{\alpha j}{}^{m} h^{kj} \tag{56}$$

From now on, let us take the cosmological constant to be zero ($\lambda = 0$). Thus, with $\kappa = \varepsilon$, the scaled version of the field equations (15) becomes

$$\bar{R}_{ij} - \tfrac{1}{2} \bar{R} \bar{g}_{ij} = \varepsilon \bar{\tau}_{ij}$$

Remembering that the curvature tensor vanishes in the zeroth-order approximation, we see that the first-order approximation of (15) is given by

$$R_{1ij} - \tfrac{1}{2} R_1 h_{ij} = \tau_{0ij} \tag{57}$$

Writing this out explicitly, we have

$$\partial_{[m} W_{0i]}^{\alpha} l_{\alpha j}{}^{m} - \tfrac{1}{2} h_{ij} \partial_{[m} W_{0p]}^{\alpha} l_{\alpha q}{}^{m} h^{pq} = \tau_{0i} \tag{58}$$

The momentum-energy tensor of the corresponding classical gauge theory thus drives the first-order approximation for the compensating fields of the Lorentz sector!

The spin field equations (16) are handled in a similar fashion. We first note that

$$\bar{L}_{\alpha k}^{i} = \varepsilon l_{\alpha k}^{i} + \cdots, \qquad \bar{S}_{kj}^{i} = \varepsilon S_{1kj}^{i} + \cdots \tag{59}$$

where

$$S_{1kj}^{i} = W_{0[k}^{\alpha} l_{|\alpha|j]}^{i} + \partial_{[k} B_{1j]}^{i} \tag{60}$$

When (18) is substituted into (16), the scaled spin equations become

$$\bar{B} \bar{g}^{km} (\bar{S}_{im}^{j} \bar{L}_{\alpha k}^{i} - 2\bar{S}_{im}^{i} \bar{L}_{\alpha k}^{j}) = \varepsilon^2 \bar{b}_k^j \bar{L}_A^k M_{\alpha B}^{A} \Psi^E$$

A cancellation of the resulting ε^2 terms on both sides of these equations gives the first-order approximation for the spin equations:

$$h^{km} (S_{1im}^{j} l_{\alpha k}^{i} - 2 S_{1im}^{i} l_{\alpha k}^{j}) = L_{0A}^{j} M_{\alpha B}^{A} \Psi_0{}^{E} \tag{61}$$

The spin currents of the classical gauge theory of mater fields on Minkowski space thus drive the spin field equations in first approximation. On the other hand, if we view (61) as a system of equations for the determination of the leading terms in torsion (remember that the torsion tensor starts with terms of order ε), we see that the spin currents of the classical gauge theory of matter serve to determine the leading terms in torsion. In particular, this determination is algebraic.

For the matter field equations, we go back to the form given by (28). A lengthy, but straightforward analysis similar to that used above gives the following first-order approximation for the matter fields:

$$
\begin{aligned}
&\partial_j L_{1A}^{j} - A_{0j}^{b} f_{bA}^{E} L_{1E}^{j} - L_{1A} - A_{1j}^{b} f_{bA}^{E} L_{0E}^{j} \\
&\quad = -\partial_j \{ B_{1i}^{i} L_{0A}^{j} + b_{1i}^{j} L_{0A}^{i} \} + A_{0j}^{b} f_{bA}^{E} (B_{1i}^{i} L_{0E}^{j} + b_{1i}^{j} L_{0E}^{i}) \\
&\quad\quad + W_{0j}^{\alpha} M_{\alpha A}^{E} L_{0E}^{j} + B_{1i}^{i} L_{0A}
\end{aligned} \tag{62}
$$

The first-order gravitational corrections to the matter fields might be expected to be complicated, and indeed they are, as (62) boisterously shows. Other than to note that the same differential operator occurs on the left-hand sides of (49) and (62), there is not much more that can be said without an explicit choice for the Lagrangian for the matter fields.

Proceeding in exactly the same way, one can obtain the first-order approximation of the field equations for the compensating 1-forms for the internal symmetry group as

$$
\begin{aligned}
&2\partial_j G_{1b}^{ji} - 2A_{0j}^{a} k_{a}{}^{c}{}_{b} G_{1c}^{ji} - 2A_{1j}^{a} k_{a}{}^{c}{}_{b} G_{0c}^{ji} - L_{1A}^{i} f_{bE}^{A} \Psi_0{}^{E} - L_{0A}^{i} f_{bE}^{A} \Psi_1{}^{E} \\
&\quad = B_{1k}^{k} L_{0A}^{i} f_{bE}^{A} \Psi_0{}^{E} + b_{1m}^{i} L_{0A}^{m} f_{bE}^{A} \Psi_0{}^{E}
\end{aligned} \tag{63}
$$

There is not much that can be said here, except to note that the compensating fields for the internal symmetry group have nontrivial first-order gravitational corrections.

The field equations for the first-order correction fields are (57) and (61)-(63). Of these, (57) and (61) are seen to be *linear* field equations that can be solved by themselves; that is, they do not involve the first-order Ψ fields or A fields. In fact, they serve to determine the zeroth-order W fields and ϕ fields (remember that these fields always occur multiplied by ε, so that the zeroth-order W and ϕ fields determine the first-order geometric quantities). This is simply a reflection of the fact that the P_{10} field equations are at least of first order in ε. Once the zeroth-order W fields and ϕ fields have been determined, they may be substituted into (62) and (63), which become *linear* field equations for the determination of the first-order Ψ fields and A fields. We thus have field equations for all first-order correction fields and the second part of the acid test is satisfied.

An inspection of the first-order field equations shows that a specific problem obtains only upon specification of the matter Lagrangian and the free field Lagrangian for the compensating fields for the internal symmetry group. Further progress can thus be made only for explicit problems. The third part of the acid test cannot be implemented except in the context of specific problems. We can only note that the representation

$$
\Psi^{A} = \Psi_0{}^{A} + 8\pi G c^{-4} \Psi_1{}^{A} + \cdots \tag{64}
$$

indicates that changes in the observables due to the first-order terms may be expected to be negligible except in situations where the first-order fields are large on the scale determined by ε.

The accuracy or lack of accuracy that obtains in the first-order approximation cannot be upset by a change of guage for P_{10}, because the ε-expansion process has already broken the local gauge covariance. The coordinate cover of M_4 and the gauge are fixed when we solve the zeroth-order field equations, and whatever then obtains in the first-order approximation has to be lived with. It cannot be changed after the fact in order to improve the approximation!

It is interesting to note at this juncture that breaking of the P_{10}-gauge covariance by the ε-expansion process has features in common with spontaneous symmetry breaking in classical gauge theory. Noting that the zeroth-order theory is classical gauge theory on Minkowski space-time, we may view the ε-expansion process as an expansion about solutions of classical gauge theory. Indeed, (64) may be rewritten in the equivalent form

$$\Psi^A = \Psi_0{}^A + \Xi^A \tag{65}$$

in which case Ξ^A represent the total correction to the Minkowski values. We can then proceed to find field equations for the Ξ's by substituting (65) and corresponding expressions for the other field variables into the original field equations. The process is thus seen to be dependent on the specific ε expansion used above only in that the zeroth-order approximation in ε is classical gauge theory on Minkowski space-time. Now, classical gauge theory obtains in Minkowski space-time, and hence expansion about solutions of classical gauge theory forces P_{10} to act only globally. In effect, looking for solutions in a neighborhood of a solution of classical gauge theory breaks the local action of P_{10} all the way down to global action. There, is thus an essential difference between the symmetry breaking involved in direct gauge theory for P_{10} and spontaneous symmetry breaking in classical gauge theory, but then the group actions are also essentially different—the internal symmetry group acts only on the matter fields, while P_{10} acts both on the matter fields and on the base manifold. It is then easily seen that it is the action of P_{10} on the base manifold that is the cause of the violent symmetry breaking from local action to global action when expanding about solutions of the classical theory.

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